



MATHEMATICS SPECIALIST : UNITS 1 & 2, 2023

Tuesday 8th August (T3W4)

Test 3 – Scalar Product and Trigonometry (10%)

1.3.10 – 1.3.14; 1.1.16; 2.1.1 – 2.1.9

Time Allowed 25 minutes	First Name	Surname	Marks / 24 marks
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Circle your Teacher's Name:

Ms Chua

Mr Koulianos

Mr Liu

Ms Robinson

Assessment Conditions: (N.B. Sufficient working out must be shown to gain full marks)

- ❖ Calculators: Not Allowed
- ❖ Formula Sheet: Provided
- ❖ Notes: Not Allowed

PART A – CALCULATOR FREE

Question 1 (2 marks)

Consider the vectors $\underline{a} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} -6 \\ 2 \end{pmatrix}$. Prove that $\underline{a} + \underline{b}$ is not perpendicular to $\underline{c}$.

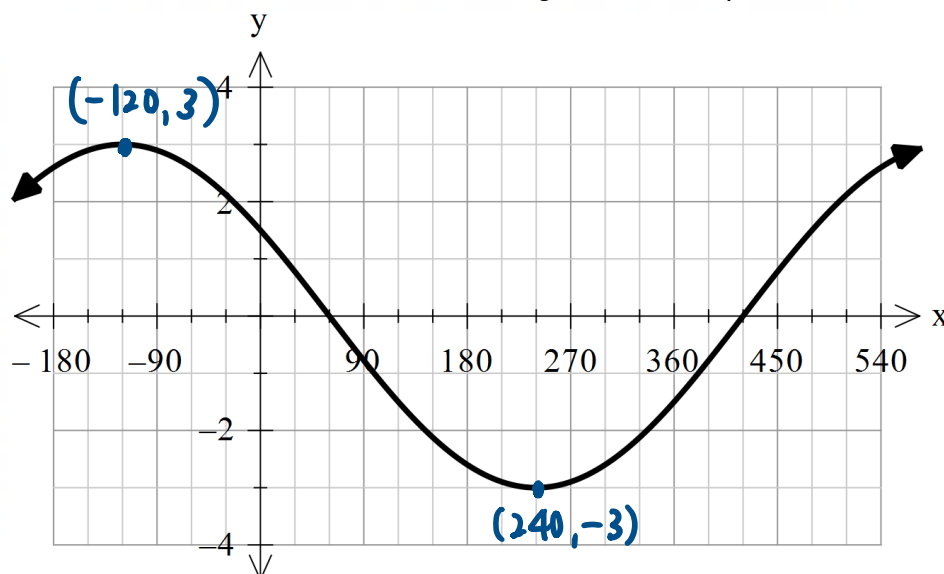
$$\begin{aligned} \underline{a} + \underline{b} &= \begin{pmatrix} 5 \\ -4 \end{pmatrix} \\ (\underline{a} + \underline{b}) \cdot \underline{c} &= \begin{pmatrix} 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \end{pmatrix} \\ &= -30 - 8 \\ &= -38 \end{aligned}$$

$\therefore$ Since $(\underline{a} + \underline{b}) \cdot \underline{c} \neq 0$,
 $\underline{a} + \underline{b}$ is not $\perp$ to $\underline{c}$.

- ✓ determines value of $(\underline{a} + \underline{b}) \cdot \underline{c}$
- ✓ states $\neq 0$ to justify not perpendicular

Question 2 (3 marks)

The equation of the graph below is in the form $y = -a \sin[b(x^\circ - c^\circ)]$, where a , b and c are positive. Determine the values of a , b and c , stating the smallest possible value for c .



Amplitude = 3 = a
 Period = $720^\circ = \frac{360^\circ}{b}$
 $\therefore b = \frac{1}{2}$
 Horizontal translation
 right 60°
 $\Rightarrow c = 60^\circ$

- ✓ correct a
- ✓ correct b
- ✓ correct c

Question 3 (6 marks = 2, 2, 2)

Vectors x and y are such that $|x| = 5$, $|y| = 4$ and $x \cdot y = 10\sqrt{3}$.

a) Determine the angle between x and y .

$$\underline{x} \cdot \underline{y} = |\underline{x}| \cdot |\underline{y}| \cdot \cos \theta$$

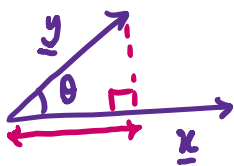
✓ substitutes into scalar product to form equation

$$10\sqrt{3} = 5 \times 4 \times \cos \theta$$

✓ solves for angle (both degrees or radians ok)

$$\cos \theta = \frac{\sqrt{3}}{2} \quad \therefore \text{angle between } \underline{x} \text{ \& } \underline{y} \text{ is } 30^\circ \text{ (or } \frac{\pi}{6} \text{ rad.)}$$

b) Determine the scalar projection of y onto x .



$$\text{scalar projection} = \frac{\underline{x} \cdot \underline{y}}{|\underline{x}|} = |\underline{y}| \cos \theta$$

$$= \frac{10\sqrt{3}}{5} \quad (\text{or } 4 \times \cos 30^\circ)$$

$$= 2\sqrt{3}$$

✓ substitutes into either formula

✓ simplified scalar projection

Answer only 1 mark

c) Determine $(x + 2y) \cdot (x - y)$.

$$(\underline{x} + 2\underline{y}) \cdot (\underline{x} - \underline{y}) = |\underline{x}|^2 + 2\underline{x} \cdot \underline{y} - \underline{x} \cdot \underline{y} - 2|\underline{y}|^2$$

$$= 25 + 10\sqrt{3} - 2 \times 16$$

$$= -7 + 10\sqrt{3}$$

✓ expands using algebraic properties (ok if left as $x \cdot x$ etc.)

✓ simplified evaluation

Deduct 1 mark for incorrect or missing vector notation

Question 4 (7 marks = 4, 3)

a) Solve $\sin 3x - \sin x = \cos 2x$ for $0 \leq x \leq 2\pi$.

$$2 \cos \left(\frac{3x+x}{2} \right) \sin \left(\frac{3x-x}{2} \right) = \cos 2x$$

$$2 \cos 2x \sin x - \cos 2x = 0$$

$$\cos 2x (2 \sin x - 1) = 0$$

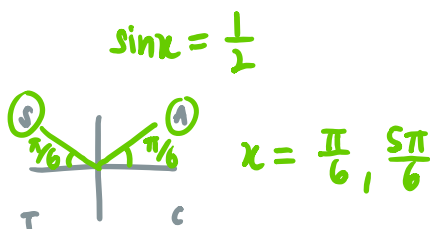
$$\cos 2x = 0 \quad 0 \leq 2x \leq 4\pi$$

$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

- ✓ rewrites using sum to product rule
- ✓ factorises and determines both equations
- ✓ solves for one set
- ✓ solves for the other set

If divide by $\cos 2x$ instead of factorise, max 2 marks

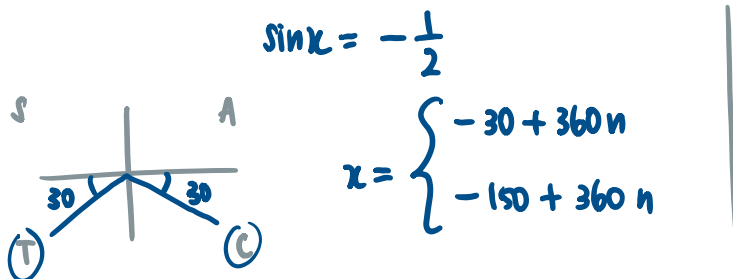


$$\therefore x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

b) In terms of $n \in \mathbb{Z}$, state the general solution to $2 \sin^2 x^\circ = \sin x^\circ + 1$.

$$2 \sin^2 x - \sin x - 1 = 0$$

$$(2 \sin x + 1)(\sin x - 1) = 0$$



$$\sin x = 1$$

$$x = 90 + 360n$$

$$\therefore x = \begin{cases} 90 + 360n \\ -30 + 360n \\ -150 + 360n \end{cases}$$

- ✓ factorises quadratic
- ✓ solves for $\sin x = -0.5$
- ✓ solves for $\sin x = 1$

Allow either set of solutions

$$\text{or } x = \begin{cases} 90 + 360n \\ 210 + 360n \\ 330 + 360n \end{cases}$$

Question 5 (3 marks)

In terms of $n \in \mathbb{Z}$, state the general solution to $2 \sec^2 4\theta + \tan^2 4\theta = 3$, where θ is in radians.

$$2(\tan^2 4\theta + 1) + \tan^2 4\theta = 3$$

$$3 \tan^2 4\theta = 1$$

$$\tan^2 4\theta = \frac{1}{3}$$

$$\tan 4\theta = \pm \frac{1}{\sqrt{3}}$$

✓ uses identity to simplify equation to this line

$$\Rightarrow 4\theta = \begin{cases} \frac{\pi}{6} + \pi n \\ -\frac{\pi}{6} + \pi n \end{cases}$$

✓ solves correctly for 4theta

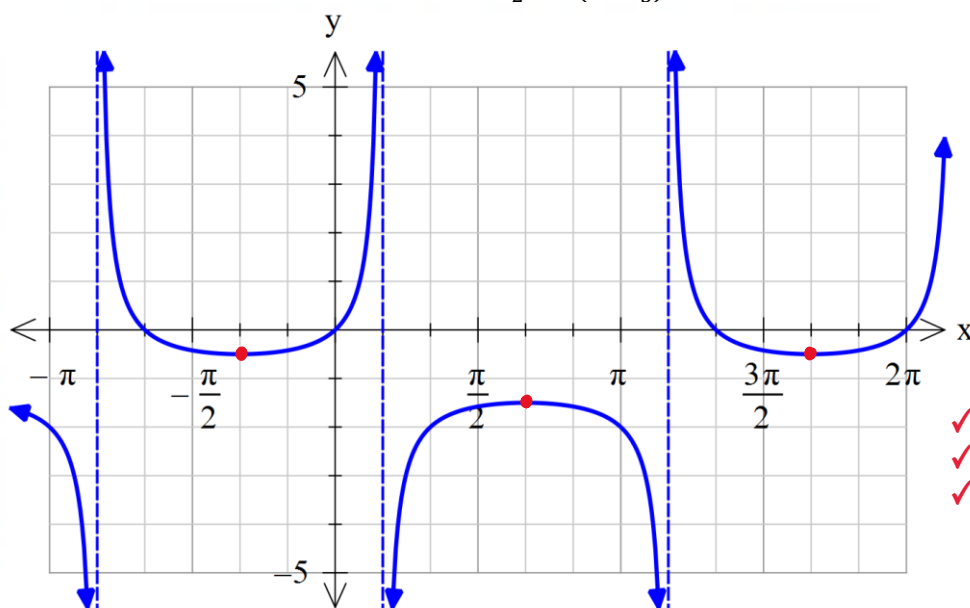
✓ solves correctly for theta

$$\therefore \theta = \begin{cases} \frac{\pi}{24} + \frac{\pi}{4}n \\ -\frac{\pi}{24} + \frac{\pi}{4}n \end{cases}$$

Allow full f/t if missing the +/- in previous working

Question 6 (3 marks)

On the axes below, sketch the graph of $y = \frac{1}{2} \sec\left(x + \frac{\pi}{3}\right) - 1$, showing clearly all key features.



✓ shows all asymptotes

✓ all min/max correct

✓ smooth curves with correct location/orientation etc.



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PART B – CALCULATOR ASSUMED

Question 7 (5 marks = 3, 2)

- a) Express $2 \cos \theta - 3 \sin \theta$ in the form $r \cos(\theta + \alpha)$ with exact r , and acute α in radians, correct to two decimal places.

$$r = \sqrt{2^2 + 3^2} = \sqrt{13} \Rightarrow \sqrt{13} \left(\cos \theta \times \frac{2}{\sqrt{13}} - \sin \theta \times \frac{3}{\sqrt{13}} \right)$$

$$\cos \alpha = \frac{2}{\sqrt{13}} \Rightarrow \alpha = 0.98 \text{ (2dp)}$$

$$\therefore 2 \cos \theta - 3 \sin \theta = \sqrt{13} \cos(\theta + 0.98)$$

- ✓ solves r
- ✓ sets up equation for α
- ✓ correct final expression with correct rounding

Answer only 2 marks

- b) Hence, determine the minimum value of $2 \cos \theta - 3 \sin \theta$ and the first positive value of θ for which it occurs.

$$\text{min. value} = -\sqrt{13}$$

$$\text{occurs first at } \theta = 2.16 \text{ rad (2dp)}$$

- ✓ correct minimum
- ✓ correct angle (reasonable rounding)

Question 8 (3 marks)

Consider the vectors $x = \begin{pmatrix} a \\ b \end{pmatrix}$, $y = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $z = \begin{pmatrix} 14 \\ -5 \end{pmatrix}$.

Determine the values of a and b given that x and y are perpendicular, and $x + z$ is parallel to y .

$$x \perp y$$

$$\Rightarrow x \cdot y = 0$$

$$3a - 2b = 0 \quad (1)$$

$$x + z = \begin{pmatrix} a+14 \\ b-5 \end{pmatrix}$$

$$(x+z) \parallel y \Rightarrow x+z = \lambda y$$

$$a+14 = 3\lambda \quad (2)$$

$$b-5 = -2\lambda \quad (3)$$

$$\text{CAS solve } (1), (2) \text{ \& } (3) \text{ or } (1) \text{ \& } (4)$$

$$\Rightarrow a = -2 \text{ \& } b = -3 \quad (\lambda = 4)$$

- ✓ sets up equation for perpendicular
- ✓ sets up equation/s for parallel (ok if use ratio or alternate ver.)
- ✓ solves both a and b

$$\hookrightarrow \text{ALTERNATE: } (x+z) \perp x$$

$$(x+z) \cdot x = 0$$

$$\Rightarrow a(a+14) + b(b-5) = 0 \quad (4)$$

solving with the above will also give a second soln $a=b=0$

Deduct if this soln is kept
No deduction if only state valid soln

Question 9 (7 marks = 3, 4)

Prove each of the following.

a) $\frac{\cos t}{1-\sin t} = \frac{1+\sin t}{\cos t}$

$$\text{LHS} = \frac{\cos t}{1-\sin t} \times \frac{1+\sin t}{1+\sin t} \quad \checkmark$$

$$= \frac{\cos t (1+\sin t)}{1-\sin^2 t}$$

$$= \frac{\cos t (1+\sin t)}{\cos^2 t} \quad \checkmark$$

$$= \frac{1+\sin t}{\cos t} \quad \checkmark$$

$$= \text{RHS}$$

For all trigonometric proof questions,
allow full marks if students have clearly shown their working
to rewrite both sides of the equation SEPARATELY to achieve
the same expression, then conclude LHS=RHS.

- ✓ multiplies with 'conjugate' of denominator
- ✓ applies Pythagorean identity to simplify denominator
- ✓ clearly simplifies to complete proof

b) $\frac{\sin 4t - \sin 2t}{\cos 4t + \cos 2t} = \tan t$

$$\begin{aligned}
 \text{LHS} &= \frac{\sin(2 \times 2t) - \sin 2t}{\cos(2 \times 2t) + \cos 2t} \\
 &= \frac{2\sin 2t \cos 2t - \sin 2t}{2\cos^2 2t - 1 + \cos 2t} \quad \checkmark \\
 &= \frac{\sin 2t (2\cos 2t - 1)}{(2\cos 2t - 1)(\cos 2t + 1)} \quad \checkmark \\
 &= \frac{2\sin 2t \cos 2t}{2\cos^2 2t - 1 + 1} \quad \checkmark \\
 &= \frac{2\sin 2t \cos 2t}{2\cos^2 2t} \\
 &= \frac{\sin 2t}{\cos 2t} \quad \checkmark \\
 &= \tan 2t \\
 &= \text{RHS}
 \end{aligned}$$

- ✓ applies double angle identity to (4t) angles
- ✓ factorises and simplifies
- ✓ applies double angle identity to (2t) angles
- ✓ clearly simplifies to complete proof

Question 10 (4 marks)

Prove $\frac{\sec \theta \sin \theta}{\tan \theta + \cot \theta} = \sin^2 \theta$.

$$\begin{aligned}
 \text{LHS} &= \frac{\frac{1}{\cos \theta} \times \sin \theta}{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}} \quad \checkmark \\
 &= \frac{\frac{\sin \theta}{\cos \theta}}{\frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta}} \quad \checkmark \\
 &= \frac{\sin \theta}{\cos \theta} \div \frac{1}{\sin \theta \cos \theta} \\
 &= \frac{\sin \theta}{\cos \theta} \times \frac{\sin \theta \cos \theta}{1} \quad \checkmark \\
 &= \sin^2 \theta \quad \checkmark \\
 &= \text{RHS}
 \end{aligned}$$

- ✓ substitutes for sec, tan and cot
- ✓ rewrites denominator as one fraction
- ✓ applies Pythagorean identity and shows simplifying the division of fractions
- ✓ simplifies to complete proof

Question 11 (5 marks = 1, 1, 3)

Shohei is on holiday and takes a ride on a Ferris wheel. He uses his smart phone to measure his altitude during the ride. The equation $H = -24.5 \cos\left(\frac{\pi}{5}t\right) + 26$ can be used to model the data collected, where H is the height in metres above ground level and t is the time in minutes since the ride commenced.

a) How long does it take to complete one revolution on the Ferris wheel?

$$\text{period} = \frac{2\pi}{\pi/5} = 10 \text{ min for one rev.} \quad \checkmark \text{ correct time with units}$$

b) What is the diameter of the Ferris wheel?

$$\begin{aligned}
 \text{diam.} &= \text{dist. between max \& min} \\
 &= 2 \times 24.5 = 49 \text{ metres} \quad \checkmark \text{ correct distance}
 \end{aligned}$$

c) The ocean is not far from where the Ferris wheel is located but can only be seen from a height of at least 40 metres above the ground. For what percentage of the ride will Shohei be able to see the ocean? Give your answer correct to two decimal places.

$$-24.5 \cos\left(\frac{\pi}{5}t\right) + 26 = 40$$

$$\text{CAS} \Rightarrow t = 3.468, 6.532$$

$$\therefore \text{above 40m for } 6.532 - 3.468 = 3.064 \text{ min of one cycle}$$

$$\text{percentage} = \frac{3.064}{10} \times 100\%$$

$$= 30.64\% \text{ of the ride (2dp)}$$

- ✓ solves both t-values
- ✓ determines time above 40 m
- ✓ determines percentage